Potts model with $q = 4, 6,$ and 8 states on Voronoi-Delaunay random lattice*

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$\star$ This paper is dedicated to Dietrich Stauffer

Abstract: Through Monte Carlo simulations we study two-dimensional Potts models with $q = 4$, 6 and 8 states on Voronoi-Delaunay random lattice. In this study, we assume that the coupling factor J varies with the distance r between the first neighbors as $J(r) \propto e^{-ar}$, with $a \geq 0$. The disordered system is simulated applying the singler-cluster Monte Carlo update algorithm and reweigting technique. In this model both second-order and first-order phase transition are present depending of q values and a parameter. The critical exponents ratio β/ν, γ/ν, and $1/\nu$ were calculated for case where the second-order phase transition are present. In the Potts model with $q = 8$ we also studied the distribution of clusters sizes.

Keywords: Monte Carlo simulation, spins, networks, Ising, Potts.

Introduction

This paper deals with the Potts model with $q = 4$, 6 and 8 states on two-dimensional Voronoi-Delaunay random lattices (VDRL). These lattices have a natural disorder in their coordination number. The randomness in the lattice of statistical spin models has been studied in order to access the effect of impurities and dilutions over their critical behavior. It was conjectured by Harris [1] that the sign of the critical exponent of the specific heat, α, determines whether the system is affected or not by such a randomness. For positive values of α the impure system should have a critical behavior different from the one of the pure system. For negative values of α, on the other hand, the critical behavior of the system should be same for both cases. In two-dimensional regular lattices, the ferromagnetic Potts model with q states displays first order phase transitions for $q > 4$ [2, 3]. The pure ferromagnetic three-state Potts model has $\alpha = 1/3$, hence, from the Harris criterium we expect to find a different behavior with a random interaction system. However, Picco [4] and Lima et al. [5] studied this model with different types of disorder and did not find significant differences from the pure

case. The q-state Potts model has been studied in scale-free networks by Igloi and Turban [6] and depending on the value of q and of the degree-exponent γ first- and second-order phase transitions are found. This model was also studied by Lima [7] on *directed* Barabási-Albert(BA) networks, where only one first-order phase transition has been obtained independent of q-values for connectivity $z = 2$ and $z = 7$ of the *directed* BA network. Here, we studied the Potts model with $q = 4$, 6, and 8 states. We also calculate the critical exponents ratio β/ν, γ/ν and $1/\nu$ for second-order phase transitions raised by disorder of the VDRL.

Model and simulation

The Voronoi construction or tessellation for a given set of points in the plane is defined as follows [8]. Initially, for each point one determines the polygonal cell consisting of the region of space nearer to that point than to any other point. Then one considers that the two cells are neighboring when they possess a boundary in common. From the Voronoi tessellation the dual lattice can be obtained by the following procedure:

(*a*) when two cells are neighbors, a link is placed between the two points located in the cells;

(*b*) From the links one obtains the triangulation of space that is called the Delaunay lattice;

(*c*) The Delaunay lattice is dual to the Voronoi tessellation in the sense that points corresponding to cells link to edges, and triangles to the vertices of the Voronoi tessellation.

The Hamiltonian of an q-states ferromagnetic Potts model can be written as

$$- H = \sum_{<i,j>} J_{ij} \delta_{\sigma_i \sigma_j}, \tag{1}$$

where δ is the Kronecker delta function, the sum goes over all nearest-neighbors pairs of sites and the spin σ can take the values $\sigma = 1, ..., q$. Here we assume that the coupling factor J_{ij} depends on the relative distance r_{ij} between sites i and j according to the following expression

$$J_{ij} = J_0 e^{-ar_{ij}}, \tag{2}$$

where J_0 is a constant, set equal to unity for simplicity, and $a \geq 0$ is a model parameter.

The simulations have been performed applying the single-cluster update algorithm [9] on different lattice sizes comprising a number $N = 250$, 1000, 2000, 4000, and 8000 of sites. For simplicity, the length of the system is defined here in terms of the size of a regular lattice $L = N^{1/2}$. For each system size quenched averages over the connectivity disorder are approximated by averaging over $R = 20$ independent realizations. For each simulation we have started with a uniform configuration of spins (the results are however independent of the initial configuration). We ran 2.52×10^6 Monte Carlo steps (MCS) per spin with 1.2×10^5 configurations discarded for thermalization using the "perfect" random-number generator [10].

In studying the critical behavior of the model using the single-cluster algorithm we define the variable $e = E/N$, where E is the energy of system, and the magnetisation of system $M = (q. \max[n_i] - N)/(q-1)$ in a time series file, where $n_i \leq N$ denote the number of spins with "orientation" $i = 1, ..., q$. From the fluctuations of e measurements we can compute: the average of e, the specific heat C and the fourth-order cumulant of e,

$$u(K) = [< E >]_{av}/N, \tag{3}$$

$$C(K) = K^2 N[< e^2 > - < e >^2]_{av}, \tag{4}$$

$$B(K) = [1 - \frac{< e^4 >}{3 < e^2 >^2}]_{av}, \tag{5}$$

where $K = 1/k_B T$, T is the temperature, and k_B is the Boltzmann constant. Similarly, we can derive from the magnetization measurements the average magnetization ($m = M/N$), the susceptibility, and the magnetic cumulants,

$$m(K) = [< |m| >]_{av}, \tag{6}$$

$$\chi(K) = KN[< m^2 > - < |m| >^2]_{av}, \tag{7}$$

$$U_4(K) = [1 - \frac{< m^4 >}{3 < m^2 >^2}]_{av}. \tag{8}$$

where $< ... >$ stands for a thermodynamic average and $[...]_{av}$ square brackets for an average over the 20 realizations.

In order to verify the order of the transition for this model, we apply finite-size scaling (FSS) [13]. Initially we search for the minima of the fourth-order parameter of Eq. (4). This quantity gives a qualitative as well as a quantitative description of the order of the transition [14]. It is known

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[15] that this parameter takes a minima value $B_{\min}$ at effective transition temperature $T_c(N)$. One can show [16] that for a second-order transition $\lim_{N \to \infty} (2/3 - B_{\min}) = 0$, even at T_c, while at a first-order transition the same limit measuring the same quantity is small and $(2/3 - B_{\min}) \neq 0$.

A more quantitative analysis can be carried out through the FSS of the C fluctuation $C_{\max}$, the susceptibility maxima $\chi_{\max}$ and the minima of the Binder parameter $B_{\min}$.

If the hypothesis of a first-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form [17],

$$C_{\max} = a_C + b_C N + ... \tag{9}$$

$$\chi_{\max} = a_\chi + b_\chi N + ... \tag{10}$$

$$B_{\min} = a_B + b_B/N + ... \tag{11}$$

Therefore, if the hypothesis of a second-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form

$$C = C_{reg} + L^{\alpha/\nu} f_C(x)[1 + ...], \tag{12}$$

$$m = L^{-\beta/\nu} f_m(x)[1 + ...], \tag{13}$$

$$\chi = L^{\gamma/\nu} f_\chi(x)[1 + ...], \tag{14}$$

$$\frac{dU_4}{dT} = L^{1/\nu} f_U(x)[1 + ...], \tag{15}$$

$$|\frac{d \ln m^p}{d K}|_{\max} = |\frac{dm^p/dK}{m^p}|_{\max} \propto L^{p/\nu}[1 + ...], \tag{16}$$

where C_{reg} is a regular background term, ν, α, β, and γ are the usual critical exponents, and $f_i(x)$ are FSS functions with

$$x = (K - K_c)L^{1/\nu} \tag{17}$$

being the scaling variable, and the brackets $[1 + ...]$ indicate corretions-to-scaling terms. Therefore, from the size dependence of M and χ we obtained the exponents β/ν and γ/ν, respectively. The exponent $1/\nu$ is obtained from the relations (15) and (16). The maxima value of susceptibility also scales as $L^{\gamma/\nu}$.
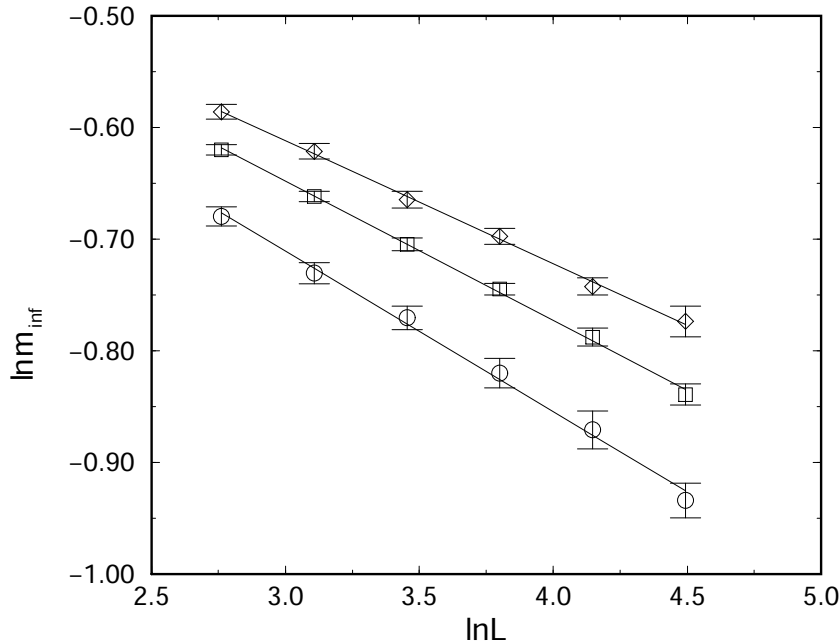
Figure 1: Logarithmic plot of the magnetisation at the inflection point versus the size system L for $a = 0.0$ (circles), 0.5 (squares) and 1.0 (diamonds), $q = 4$.

Cluster distributions in two-dimensional 8-states Potts model on VDRL

To distinguish the order of the phase transition is one of various problems of Monte Carlo simulations for spin systems. The greatest difficulty arises when the correlation length is finite and larger than the size of the system.. In such situations, common tools identification of the order of transition as for example, by examining the minimum free energy [21, 22] by considering the probability distribution of energy [23] may not be in this case, a good indicator. Even in the presence of meta-stable states, the size of the system can prevent the observation of the double peak characteristic of the first-order phase transition, in the distribution of energy. The basic reason for such behavior occurs because the fact that energy is a local quantity. Quantities of a global nature are expected to be more sensitive to the correlation length and so the effects of meta-stable states become more evident.

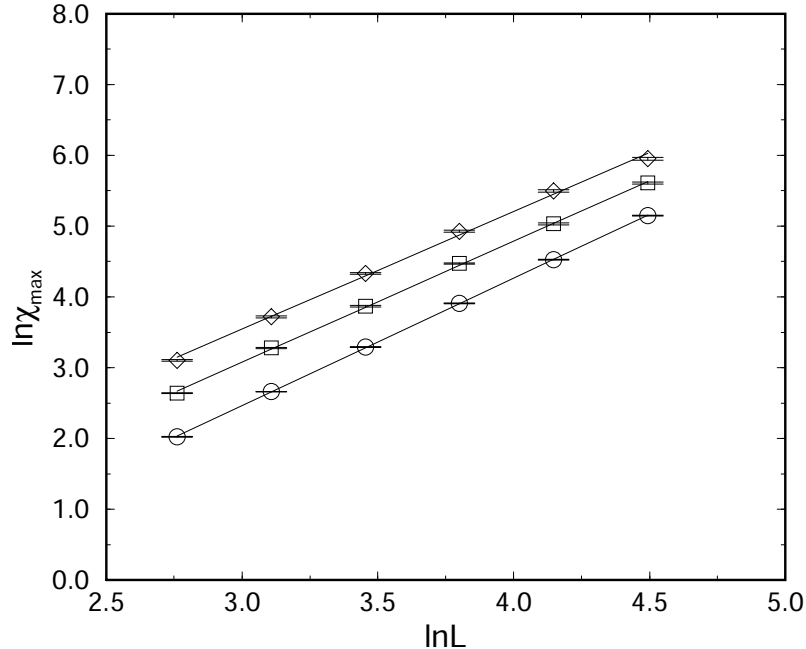If a system undergoes a second-order phase transition, the correlation

Figure 2: Logarithmic plots of the maximum suscepbility versus the size system L for $a = 0$ (circles), 0.5 (squares) and 1.0 (diamonds), $q = 4$.

length is infinitely large at the critical point of the system which results in the formation of infinitely large clusters, such that close to the point of a critical second-order phase transition there is co-existence of clusters of all sizes.

In systems that exhibit first-order phase transitions the most common feature is the coexistence of ordered and disordered states in the region of phase transition. While structures dominated by large clusters are ordered phases, small clusters are representative of the disordered phase. The existence of both small and large clusters at a first-order phase transition gives rise to the behavior of double peaks in the energy distribution.

Previous works [20, 24] on the Potts model in two dimensions observed that global operators related to clusters size are more sensitive structural changes in a phase transition that operators of sites related to energy and the order parameter of the system. Particularly the cluster distribution may give a better indication of the order of phase transition for small network sizes
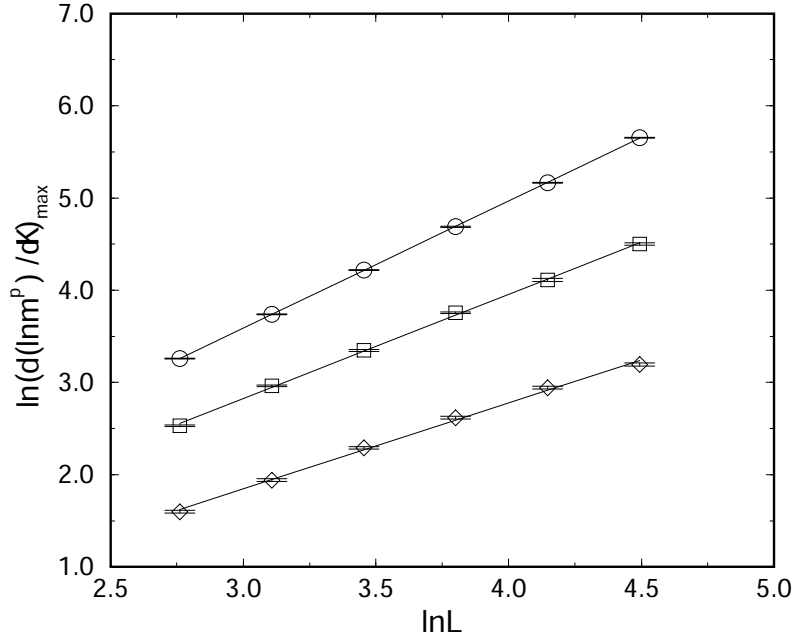
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Figure 3: Plots of scale of finite size of the maximums of the Logarithmic derivatives of $|M|^p$ versus $\ln L$ for $p = 2$ and $a = 0$ (circles), 0.5 (squares) and 1.0 (diamonds), $q = 4$.

than the energy distribution. The average clusters size S may be defined [25] as

$$S = \frac{1}{N_c} \langle \sum_{i=1}^{N_c} C_i \rangle, \tag{18}$$

where N_C is the number of clusters and C_i the spin number in the cluster i normalized by the total spins number. One possible cluster related measure is the cluster size distribution (CSD), which is evaluated in the same way as the standard energy probability distribution in the form of histogram. Another relevant quantity is the configuration average cluster size distribution (ASCD). Instead of counting all clusters, ASCD is obtained by considering average cluster sizes for each configuration.

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	ν	α/ν	β/ν	γ/ν
(i)	2/3	1	1/8	7/4
	(0.6666)	(1)	(0.125)	(1.75)
(ii) $a = 0$	0.725(2)	0.713(4)	0.143(9)	1.799(6)
	0.727(6)			
$a = 0.5$	0.88(1)	0.43(2)	0.12(2)	1.70(2)
	0.88(2)			
$a = 1.0$	1.07(3)	0.22(1)	0.10(2)	1.66(4)
	1.07(2)			

Table 1: Ferromagnetic of Potts with 4 states in two-dimensions : (i) analytical results, (ii) our results for $a = 0.0$, 0.5 and 1.0. Where have two values for ν exponent calculated using the eq. (16) for $p = 1$ and 2.

Results and Discussion

We study the critical behavior of the Potts model on VDRL for three values of a ($a = 0.0$, 0.5 and 1.0) and three values q ($q = 4$, 6, and 8). For each value of a and q, we apply the finite size scaling technique [13] together with the single-histogram algorithm. We perform the same procedure for systems with different number of sites $N = 250$, 500, 1000, 2000, 4000, and 8000. The critical temperature for infinite size system is estimated by using the fourth-order magnetization (Binder) cumulant.

In Figs. 1 to 3 we display the scalings for natural logarithm for the dependence of the magnetization $|m|$ on inflection point at $K = K_c(L)$, maximum amplitude, $\chi_{\max}(L)$, and logarithmic derivatives of $|M^p|(p = 2)$ versus $\ln L$ for $a = 0.0$ (circles), 0.5 (squares) and 1.0 (diamonds), respectively, for $q = 4$. Using the fourth-order magnetisation Binder cumulant we find for $q = 4$ the critical values $K_c = 0.6691(3)$, 1.2092(5) and 2.1353(4) and $U^* = 0.6031(2)$, 0.6199(4) and 0.6272(3), corresponding to $a = 0.0$, 0.5 and 1.0, respectively. For $q = 4$ the specific-heat exponent $\alpha = 2/3$ is good candidate to check whether a change of critical behavior can be induced at all by VDRL. Here, our results, see table 1, presents no *reliable* indication for change and in favor of the Harris criterium, error bars are only statistical, and much larger systems might give different exponents, also, the scaling laws $2 - \alpha = \gamma + 2\beta$ $= d\nu$ are partially violated, making our estimates less reliable.

For the Potts model with 6-states and $a = 0$ ($K_c = 0.7633(6)$) our simulations indicate that the model display a first-order phase transition, in perfect
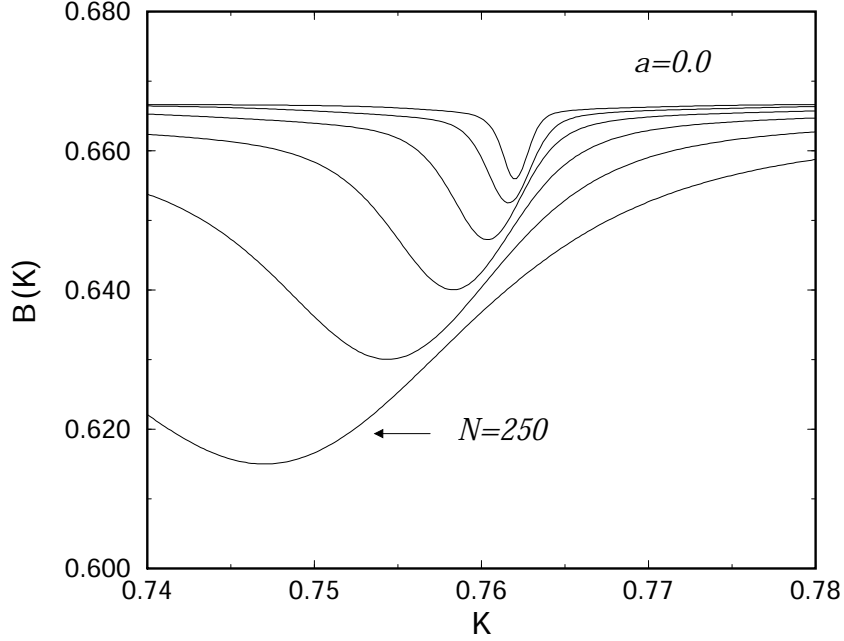
Figure 4: Plots of the Binder parameter $B(K)$ versus K for $a = 0$ and several lattices sizes ($N = 250, 500, 1000, 2000, 4000,$ and 8000) sites, for $q = 6$.

agreement with the results reported by Janke et al. [17]. However at high value of a, $a = 0.5$ and 1.0, we observe a typical second-order phase transition where the critical values $K_c = 1.3671(5)$ and $2.3972(6)$ and $U^* = 0.6236(6)$ and $0.6324(5)$, corresponding to $a = 0.5$ and 1.0, respectively, was obtained using the eq. 8.

In Figs. 4 and 5 we display the plot $B(K)$ versus K for $a = 0.0$ and 1.0 and different size lattices ($N = 250$ to 8000 sites). We can see that, in the limit of large lattice size the Binder parameter goes to $2/3$ (in Fig. 5), providing a qualitative confirmation for the presence of the continuos transition in the system. For $a = 0$ (see Fig. 4), however, the Binder parameter goes to a value which is different from $2/3$. This is a sufficient condition to characterize a first-order phase transition. The order of the transitions can be confirmed by plotting the values of $2/3 - B_{\min}(K)$ versus $1/N$ for different values of a. While for $a = 0.5$ and 1.0 the curve goes to zero as we increase the system size, for $a = 0$ the quantity $2/3 - B_{\min}(K)$ approaches a nonvanishing value in the limit of small $1/N$ (see Fig. 6). At this point, we can assume
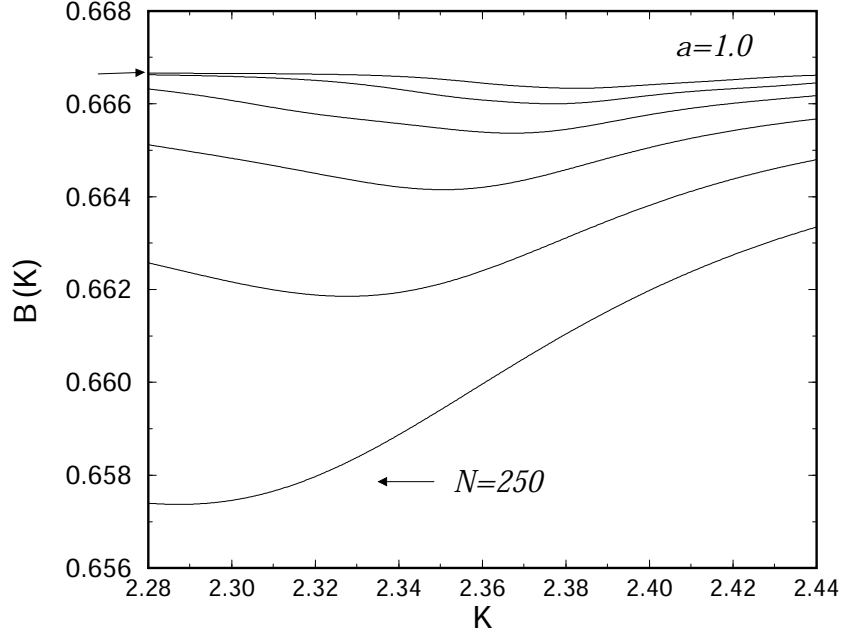
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Figure 5: The same as in Figure 4, but for $a = 1$. The arrow indicates the position of $B(K) = 2/3$.

that change in a from 0.0 to 0.5 should be followed by a crossover at a value $a = a_c$ from a first- to second-order phase transitions. A quantitative analysis can be made through of the relations (9) and (10), see Fig. 7. The exponents ratio β/ν, γ/ν and $1/\nu$ are obtained from the slopes of the straight lines for $q = 6$ and $a = 0.5$ and 1.0 (not shown here) the same way was obtained for $q = 4$, see table 2. For Potts model with $q = 8$ states Lima et al. [19] have made identical numerical analysis made here for $q = 6$, see table 3. Here we simulate, using cluster algorithms, the Potts model studied in previous sections, observing energy histograms and the ASCD in order to obtain better information on the phase transition of this model and speculate on the value of a_c mentioned previously. The cluster algorithm used here is the same Wolff algorithm used previously, except that before to calculate the observables, we studied clusters until the total number of sites in all clusters visited is equal to or greater than the total number sites of the network. I.e., while in the conventional Wolff algorithm only one cluster is formed in one Monte Carlo interaction, here we can have more than one cluster formed. In
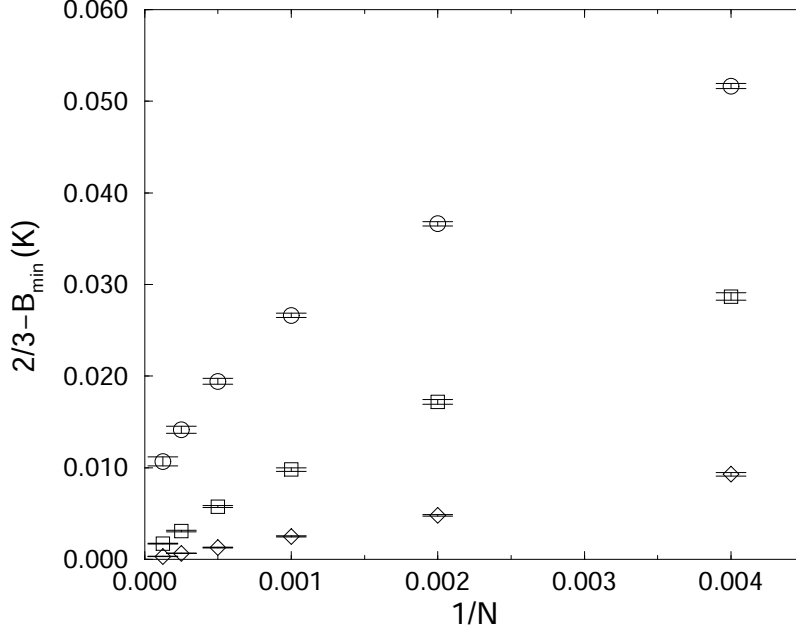
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Figure 6: Plots of $2/3 - B_{\min}(K)$ versus $1/N$ for $a = 0$ (circles), 0.5 (squares) and 1.0 (diamonds), $q = 6$.

our simulation we study the ferromagnetic Potts model with eight states at $a = 0.00, 0.10, 0.15, 0.20, 0.23, 0.25, 0.27$ and 0.50 for $N = 4000$ sites. After thermalization of 300,000 Monte Carlo steps about 10^6 steps were simulated in their respective transition points. After every two steps, we computed the energy and the average cluster size.

We examined the phase structure of the ferromagnetic Potts model with eight states with respect to quenched randomness in a range of values of a from $a = 0.00$ to 0.50. For a fixed value of a, 40 replicas were generated with different bounds distributions. Energy histograms and ASCD were obtained from averages of those obtained in the above 40 replicas generated. In Fig. 8, we show the energy histogram over a network of $N = 4000$ sites at various values of a. Starting with $a = 0.00$, which displays a first-order phase transition for a ferromagnetic Potts model with eight states studied here, we see a double peak structure in the energy histogram for values of a below $a = 0.20$. For values of a above $a = 0.25$ (x represents $a = 0.27$ and the triangles correspond to $a = 0.50$ in the Fig. 9) we observed a single Gaussian
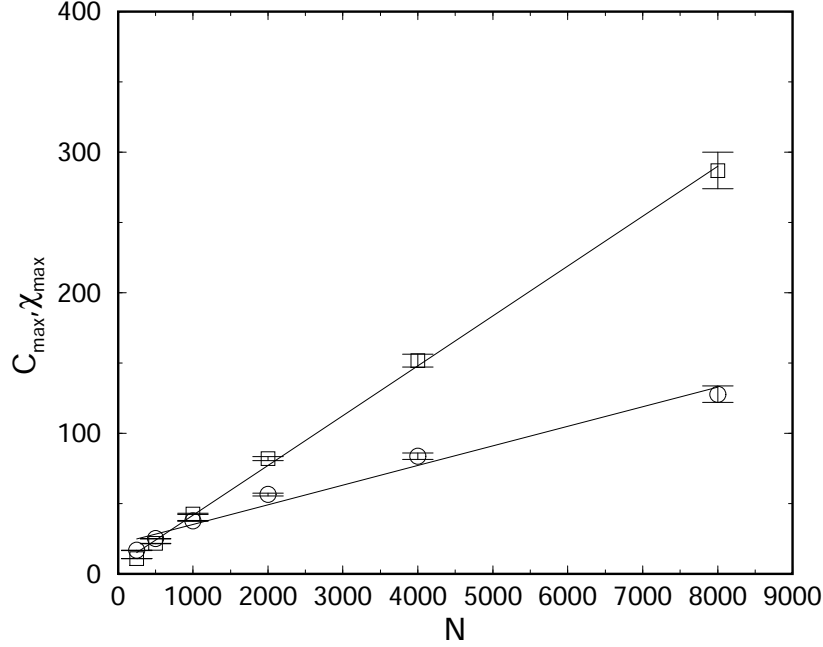
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Figure 7: Plots of the specific heat $C_{\max}$(circles) and susceptbility $\chi_{\max}$(squares) versus N for $a = 0$, $q = 6$.

peak in the energy histogram. To obtain a better signal with respect to the type of the transition, we examined the ASCD. As shown in Fig. 9, the peaks in the region of small clusters indicate the presence of first-order phase transitions. The order of the transition changes from first to second order in the range of $0.20 \leq a \leq 0.27$, where the exact value of a_c, crossover, was not obtained here, we even in this range observed the presence of meta-stable states for the size of the network studied. What remains is to verify whether the threshold value of crossover for this type of randomness is specific or not for the size of the network studied.

	ν	α/ν	β/ν	γ/ν
(i)	1	0	0.125	1.75
(ii)	1.03(3) 1.04(3)	0.182(5)	0.120(9)	1.750(6)
(iii) $a = 0.5$	0.93(5) 0.91(5)	0.51(5)	0.122(4)	1.53(5)
$a = 1.0$	1.11(4) 1.10(4)	0.22(2)	0.14(1)	1.56(5)

Table 2: (i) Analytical results for ferromagnetic Ising model $2D$, (ii) Simulation results of Janke *et. al.* [18] for ferromagnetic Ising model $2D$ on Voronoi-Delaunay random lattice. (iii) Our results for ferromagnetic $2D$ Potts model with $q = 6$ for $a = 0.5$ and 1.0. Again we have two values for exponent ν, see eq. (16) and table 1.

a	ν_1	ν_2	α/ν	β/ν	γ/ν
0.5	1.22(5)	1.17(3)	0.32(7)	0.131(7)	1.20(8)
1.0	1.26(3)	1.26(8)	0.12(3)	0.126(8)	1.45(6)

Table 3: Critical exponents for ferromagnetic Potts with 8 states in $2D$ on VRDL. Our results for $a = 0.5$ and 1.0. The ν_1 and ν_2 exponents are respectively for $p = 1$ and 2, see eq. (16).

Conclusion

In conclusion, we have presented simulations for Potts model with $q = 4$, 6 and 8 states on VDRL. The disordered system is simulated applying the singler-cluster Monte Carlo update algorithm and reweigting technique that give results with precision hight. The Potts model with $q = 4$ does display a second-order phase transition on VDRL for parameter $a = 0$, 0.5, and 1.0. For the Potts model on regular lattice the specific-heat exponent $\alpha = 2/3$ is a good candidate for a change of critical exponents on VDRL. Here, our results, summarized in table 1, presents no *reliable* indication for change, because the error bars are only statistical. Nevertheless, the present exponent estimates give a change, in particular for α/ν, and then are compatible with the Harris criterium for $N = 250, 500, 1000, 2000, 4000$, and 8000 sites used here and agree with the assumption made by Janke et al. [26] that the Harris
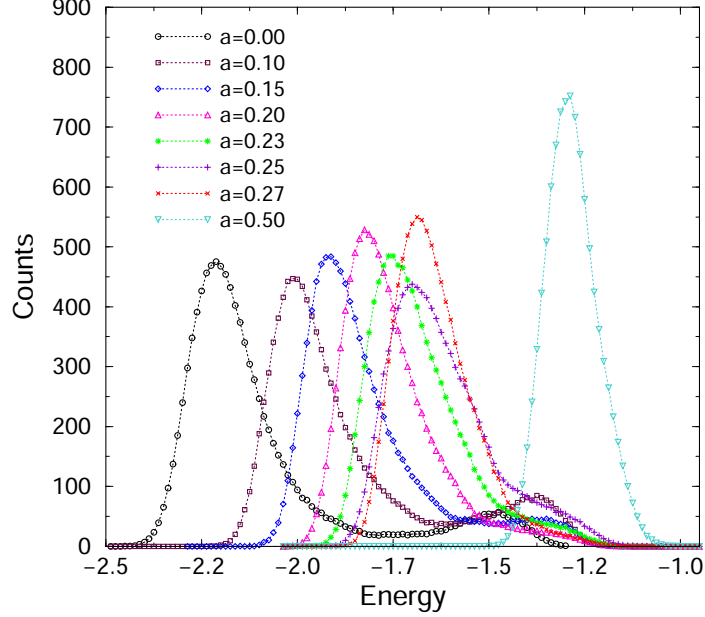
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Figure 8: Energy histogram for $a = 0.0$ (left) to 0.5 (right) and $N = 4000$ sites, $q = 8$.

criterium is not violated for $q = 4$. For $q = 6$ and 8 with $a = 0$ this model presents a first-order phase that is agreement with results for regular lattices, i.e., connectivity disorder only not is enough to change the order of phase transition that agree with results of Janke and Villanova [27]. For $a > 0$ it presents second-order phase transition. Thus, the above results, summarized in tables 2 and 3 show that the Potts model case studied here on VDRL is similar to the critical behavior of the two-dimensional $(2D)$ eight-state random-bond Potts model, S. Chen et al. [28]. They obtained the critical exponents for two sets of bond strengts, from which they concluded that the transition is the second order with critical exponents for both sets falling into same universality class, that of a $2D$ Ising model, i.e., $\alpha/\nu = 0$, $\beta = 1/8 = 0.125$, $\gamma = 7/4 = 1.75$ and $\nu = 1$. Here again the exponents ratio α/ν for $q = 6$ and 8 $(a > 0)$ are large compared to the zero for square lattice Ising model and as our disorder system is not the result of the introduction of impurities in the pure system nothing can be stated about the universality class of a
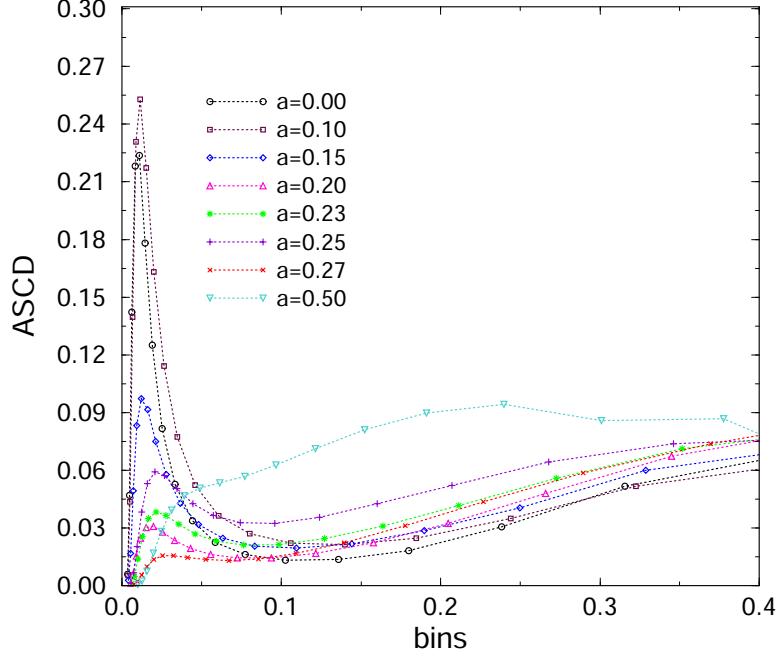
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Figure 9: Configuration averaged cluster size distribution (ASCD) for $a = 0.0$ to 0.5 and $N = 4000$ sites, $q = 8$.

$2D$ Ising model. However, the results of de Oliveira et al. [29] studying the Contact Process on a Voronoi triangulation are in disagreement with the Harris criterium; these authors suggest then that the Harris criterium must be reformulated. The exponent γ/ν for $q = 6$ and 8 states and in the cases where $a > 0$ do not change with the disorder of the Voronoi-Delaunay random lattices, but nothing can be said about the Harris criterium. Here, for Potts model with eight states we also show that using quantities depending on cluster size distribution may give complementary indications for identifying the order the phase transition for small random lattice as VRDL, which may not be derivable from energy and order parameter distribution.

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